Exercise 2.1

Q1 Differentiate to find $\frac{dy}{dx}$ for:

a) $y = x^6$ b) $y = x^3$ c) $y = x^{-2}$ d) $y = 3x^2$

e) $y = 7x$ f) $y = 3$ g) $y = 3\sqrt{x}$ h) $y = 2x^{-1}$

Q2 Differentiate to find $f'(x)$ for:

a) $f(x) = x^5$ b) $f(x) = x^7$ c) $f(x) = x^{-4}$ d) $f(x) = 4x^3$

e) $f(x) = 8\sqrt{x}$ f) $f(x) = 3\sqrt[3]{x}$ g) $f(x) = -7$ h) $f(x) = 4x^{-2}$

Q3 Find the gradient of each of the following functions:

a) $y = 2x^2$ when $x = 4$ b) $y = x^{-1}$ when $x = 2$

c) $y = -4x^5$ when $x = 1$ d) $f(x) = 2\sqrt{x}$ at the point (9, 6)

e) $f(x) = x^4$ at the point (–2, 16) f) $f(x) = -2x^3$ when $f(x) = -250$

Exercise 2.2

Q1 Differentiate these functions:

a) $y = 4x^3 - x^2$ b) $y = x + \frac{1}{x}$

c) $y = 3x^2 + \sqrt{x} - 5$ d) $f(x) = -2x^5 + 4x - \frac{1}{x^2}$

e) $f(x) = \sqrt{x^3} - x$ f) $f(x) = 5x - \frac{2}{x^3} + \sqrt[3]{x}$

Q2 Find:

a) $\frac{d}{dx}(x(x^6 - 1))$ b) $\frac{d}{dx}((x - 3)(x + 4))$

c) $\frac{d}{dx}(x(x - 1)(x - 2))$ d) $\frac{d}{dx}((x - 3)(x + 4)(x - 1))$

e) $\frac{d}{dx}(x^2(x - 4)(3 - x^3))$ f) $\frac{d}{dx}((x - 3)^2(x^2 - 2))$

Q2 Hint: Remember that $\frac{d}{dx}(\ \ )$ means the derivative with respect to x of the thing in brackets.

Q3 Find the gradient of each of the following curves:

a) $y = x^4 - x^2 + 2$ when $x = 3$ b) $y = 2x^5 + \frac{1}{x}$ when $x = -2$

c) $y = x(x - 1)(x - 2)$ when $x = -3$ d) $y = 5(x^2 - 1)(3 - x)$ when $x = 0$

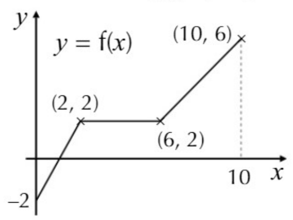
e) $y = \sqrt{x}(x - 1)$ at (4, 6) f) $f(x) = x^3(x^2 - 5)$ at (–1, 4)

g) $f(x) = \frac{1}{x^2}(x^3 - x)$ at $x = 5$ h) $f(x) = \frac{3x^3 + 18x^2 + 24x}{x + 4}$ at (–2, 0)

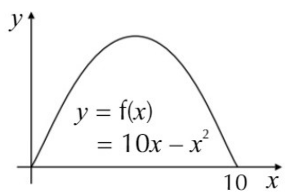
Q3h) Hint: Where there's a fraction with an expression in the denominator, try to take the denominator out of the numerator as a factor.

Q4 For the following graphs, sketch the graph of $f'(x)$ for $0 \le x \le 10$:

a)



b)



Q5 For each of the following curves, sketch the graph of $y = f'(x)$.

a) $f(x) = (x + 3)(x + 4)$ b) $f(x) = \dfrac{x^3 - 3x^2 + 2x}{x - 1}$

c) $f(x) = x^4 - 4x^3 + 4x^2 - 9$ d) $f(x) = (x - 1)^2(x + 5)$

Q6 For each of the following functions, find the coordinates of the point or points where the gradient is 0:

a) $y = x^2 - 2x$ b) $y = 3x^2 + 4x$

c) $y = 5x^2 - 3x$ d) $y = 9x - 3x^3$

e) $y = 2x^3 - x^2$ f) $y = 2x^3 + 3x^2 - 12x$

Q7 Differentiate these functions:

a) $y = \dfrac{x^2 - 3x - 4}{x + 1}$ b) $f(x) = \dfrac{x^4 - 9}{x^2 + 3}$

c) $f(x) = \dfrac{x^5 - 16x^3}{x + 4}$ d) $y = \dfrac{1}{x}(x - 3)(x - 4)$

e) $y = \sqrt{x}\left(x^3 - \sqrt{x}\right)$ f) $f(x) = \dfrac{3 - \sqrt{x}}{\sqrt{x}}$

g) $f(x) = \dfrac{x + 5\sqrt{x}}{\sqrt{x}}$ h) $f(x) = \dfrac{x - 3\sqrt{x} + 2}{\sqrt{x} - 1}$

Answers

2. Differentiating $y = f(x)$

Exercise 2.1 — Differentiating x^n

Q1 a) $\dfrac{dy}{dx} = 6x^5$ b) $\dfrac{dy}{dx} = 3x^2$

c) $\dfrac{dy}{dx} = -2x^{-3} = -\dfrac{2}{x^3}$ d) $\dfrac{dy}{dx} = 6x$

e) $\dfrac{dy}{dx} = 7$ f) $\dfrac{dy}{dx} = 0$

Exercise 2.2 — Differentiating functions

Q2

Q1 a) $\dfrac{dy}{dx} = 12x^2 - 2x$

b) $\dfrac{dy}{dx} = 1 + (-x^{-2}) = 1 - \dfrac{1}{x^2}$

c) $\dfrac{dy}{dx} = 6x + \dfrac{1}{2}x^{-\frac{1}{2}} = 6x + \dfrac{1}{2\sqrt{x}}$

d) $f'(x) = -10x^4 + 4 - (-2x^{-3}) = -10x^4 + 4 + \dfrac{2}{x^3}$

e) $f'(x) = \dfrac{3}{2}x^{\frac{1}{2}} - 1 = \dfrac{3}{2}\sqrt{x} - 1$

f) $f'(x) = 5 - 2(-3x^{-4}) + \dfrac{1}{3}x^{-\frac{2}{3}} = 5 + \dfrac{6}{x^4} + \dfrac{1}{3\sqrt[3]{x^2}}$

Q3 a) $\dfrac{dy}{dx} = 4x \Rightarrow$ At $x = 4$, $\dfrac{dy}{dx} = 16$

Q2 a) $\dfrac{d}{dx}(x(x^6 - 1)) = \dfrac{d}{dx}(x^7 - x) = 7x^6 - 1$

b) $\dfrac{d}{dx}((x - 3)(x + 4)) = \dfrac{d}{dx}(x^2 - 3x + 4x - 12)$

$= \dfrac{d}{dx}(x^2 + x - 12) = 2x + 1$

Q3 a) $\dfrac{dy}{dx} = 4x^3 - 2x$. At $x = 3$, $\dfrac{dy}{dx} = 102$.

d) $y = 5(x^2 - 1)(3 - x) = 5(-x^3 + 3x^2 + x - 3)$

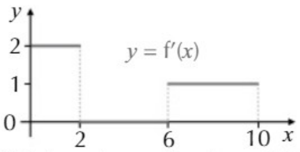
$= -5x^3 + 15x^2 + 5x - 15$

$\dfrac{dy}{dx} = -15x^2 + 30x + 5$. At $x = 0$, $\dfrac{dy}{dx} = 5$.

e) $y = \sqrt{x}(x - 1) = x^{\frac{1}{2}}(x - 1) = x^{\frac{3}{2}} - x^{\frac{1}{2}}$

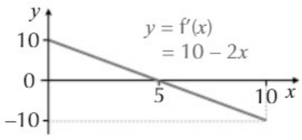
$\dfrac{dy}{dx} = \dfrac{3}{2}x^{\frac{1}{2}} - \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{3}{2}\sqrt{x} - \dfrac{1}{2\sqrt{x}}$.

Q4 a)



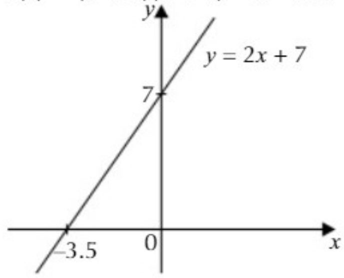
Work out the gradient for each bit of the line. You should get f′(x) = 2 for 0 ≤ x ≤ 2 and f′(x) = 1 for 6 ≤ x ≤ 10.

b)



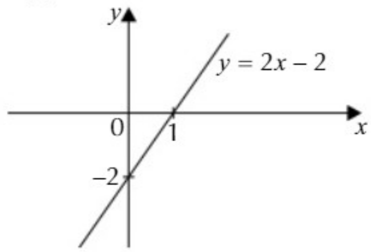
Differentiate and sketch the graph of the gradient function.

Q5 **a)** $f(x) = (x + 3)(x + 4) = x^2 + 7x + 12$, $f'(x) = 2x + 7$



b) $f(x) = \dfrac{x^3 - 3x^2 + 2x}{x-1} = \dfrac{x(x-1)(x-2)}{x-1} = x^2 - 2x$

$f'(x) = 2x - 2$



Q6 **a)** $\dfrac{dy}{dx} = 2x - 2$. If $2x - 2 = 0 \Rightarrow 2x = 2 \Rightarrow x = 1$.
So $y = (1)^2 - 2(1) = -1$. Coordinates are $(1, -1)$.

b) $\dfrac{dy}{dx} = 6x + 4$. If $6x + 4 = 0 \Rightarrow 6x = -4$
$\Rightarrow x = -\frac{4}{6} = -\frac{2}{3}$
So $y = 3(-\frac{2}{3})^2 + 4(-\frac{2}{3}) = -\frac{4}{3}$.
Coordinates are $(-\frac{2}{3}, -\frac{4}{3})$.

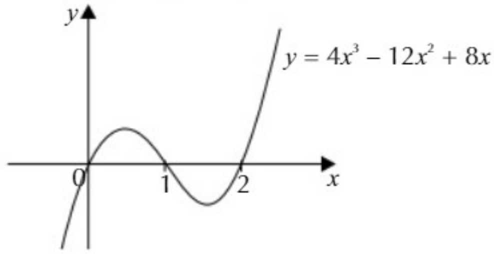
c) $\dfrac{dy}{dx} = 10x - 3$. If $10x - 3 = 0 \Rightarrow 10x = 3$
$\Rightarrow x = \frac{3}{10} \Rightarrow y = 5(\frac{3}{10})^2 - 3(\frac{3}{10}) = -\frac{9}{20}$
Coordinates are $(\frac{3}{10}, -\frac{9}{20})$.

d) $\dfrac{dy}{dx} = 9 - 9x^2$. If $9 - 9x^2 = 0 \Rightarrow 9 = 9x^2 \Rightarrow 1 = x^2$
$\Rightarrow x = 1$ or -1
$\Rightarrow y = 9(1) - 3(1)^3 = 6$ or $y = 9(-1) - 3(-1)^3 = -6$.
Coordinates of points are $(1, 6)$ and $(-1, -6)$.

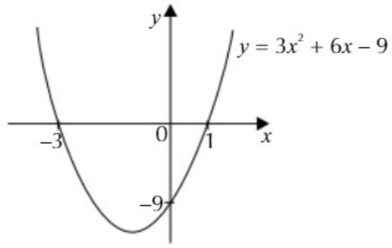
e) $\dfrac{dy}{dx} = 6x^2 - 2x$. If $6x^2 - 2x = 0 \Rightarrow 2x(3x - 1) = 0$
$\Rightarrow 2x = 0$ or $3x - 1 = 0 \Rightarrow x = 0$ or $x = \frac{1}{3}$
So $y = 2(0)^3 - (0)^2 = 0$ or $y = 2(\frac{1}{3})^3 - (\frac{1}{3})^2 = -\frac{1}{27}$.
So points are $(0, 0)$ and $(\frac{1}{3}, -\frac{1}{27})$.

f) $\dfrac{dy}{dx} = 6x^2 + 6x - 12$. If $6x^2 + 6x - 12 = 0$
$\Rightarrow 6(x^2 + x - 2) = 0 \Rightarrow x^2 + x - 2 = 0$
$\Rightarrow (x + 2)(x - 1) = 0 \Rightarrow x = -2$ or $x = 1$.
So $y = 2(-2)^3 + 3(-2)^2 - 12(-2) = 20$ or
$y = 2(1)^3 + 3(1)^2 - 12(1) = -7$.
So the points are $(-2, 20)$ and $(1, -7)$.

c) $f'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2)$
$= 4x(x - 1)(x - 2)$



d) $f(x) = (x - 1)^2(x + 5) = (x^2 - 2x + 1)(x + 5)$
$= x^3 + 5x^2 - 2x^2 - 10x + x + 5$
$= x^3 + 3x^2 - 9x + 5$
$f'(x) = 3x^2 + 6x - 9 = 3(x^2 + 2x - 3) = 3(x - 1)(x + 3)$



Q7 **a)** $y = \dfrac{x^2 - 3x - 4}{x+1} = \dfrac{(x-4)(x+1)}{x+1} = x - 4 \Rightarrow \dfrac{dy}{dx} = 1$

b) $f(x) = \dfrac{x^4 - 9}{x^2 + 3} = \dfrac{(x^2+3)(x^2-3)}{x^2+3} = x^2 - 3 \Rightarrow f'(x) = 2x$

c) $f(x) = \dfrac{x^5 - 16x^3}{x+4} = \dfrac{x^3(x+4)(x-4)}{x+4} = x^3(x - 4)$
$= x^4 - 4x^3 \Rightarrow f'(x) = 4x^3 - 12x^2$

d) $y = \frac{1}{x}(x - 3)(x - 4) = \frac{1}{x}(x^2 - 3x - 4x + 12)$
$= \frac{1}{x}(x^2 - 7x + 12) = x - 7 + \frac{12}{x} = x - 7 + 12x^{-1}$
$\Rightarrow \dfrac{dy}{dx} = 1 - 12x^{-2} = 1 - \dfrac{12}{x^2}$

e) $y = \sqrt{x}(x^3 - \sqrt{x}) = x^{\frac{1}{2}}(x^3 - x^{\frac{1}{2}}) = x^{\frac{7}{2}} - x$
$\Rightarrow \dfrac{dy}{dx} = \frac{7}{2}x^{\frac{5}{2}} - 1 = \frac{7}{2}\sqrt{x^5} - 1$

f) $f(x) = \dfrac{3 - \sqrt{x}}{\sqrt{x}} = \dfrac{3 - x^{\frac{1}{2}}}{x^{\frac{1}{2}}} = x^{-\frac{1}{2}}(3 - x^{\frac{1}{2}})$
$= 3x^{-\frac{1}{2}} - x^0 = 3x^{-\frac{1}{2}} - 1$
$f'(x) = 3(-\frac{1}{2}x^{-\frac{3}{2}}) = -\frac{3}{2}x^{-\frac{3}{2}} = -\dfrac{3}{2\sqrt{x^3}}$

g) $f(x) = \dfrac{x + 5\sqrt{x}}{\sqrt{x}} = \dfrac{x + 5x^{\frac{1}{2}}}{x^{\frac{1}{2}}} = x^{-\frac{1}{2}}(x + 5x^{\frac{1}{2}})$
$= x^{\frac{1}{2}} + 5x^0 = x^{\frac{1}{2}} + 5$
$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$

h) Factorising the numerator:
$f(x) = \dfrac{x - 3\sqrt{x} + 2}{\sqrt{x} - 1} = \dfrac{(\sqrt{x} - 2)(\sqrt{x} - 1)}{\sqrt{x} - 1}$
$= \sqrt{x} - 2 = x^{\frac{1}{2}} - 2$
$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$